Skipping analysis of various sorting methods
Know you have covered:
Selection, Insertion, Mergesort
Poll:
Quicksort
Heapsort
Radix sort
Count sort
Bucket sort

The Selection Problem - Variable Size Decrease/Conquer (Practice with algorithm analysis)

Consider the problem of finding the i^{th} smallest element in a set of n unsorted elements. This is referred to as the selection problem or the i^{th} "order statistic".

If i=1 this is finding the minimum of a set
i=n this is finding the maximum of a set
i=n/2 this is finding the median or halfway point of a set

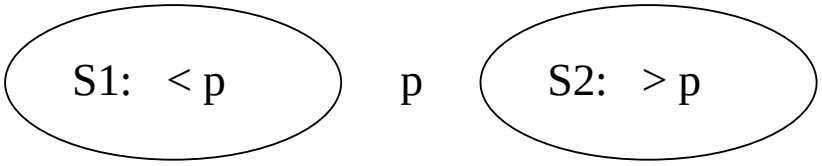
All of these scenarios are common problems.

The selection problem is defined as:

Input: A set of n numbers and a number i, with $1 \leq i \leq n$
Output: The element x in A that is larger than exactly i-1 other elements in A.

Can do in $\Theta(n \lg n)$ time easily by sorting with Merge Sort, and then pick A[i]. But we can do better!

Consider if the set of n numbers is divided as follows:



(In general we could have $\leq$ or $\geq$ but for simplicity let's just leave it as-is). Note that the elements in S1 are not sorted, but all of them are smaller than element p (partition). We know that p is the (|S1| +1)th smallest element of n. This is the same idea used in quicksort.

Now consider the following algorithm to find the i^{th} smallest element from Array A:

- Select a pivot point, p, out of array A.
- Split A into S_1 and S_2, where all elements in S_1 are < p and all elements in S_2 are > p
- If $i = |S_1|+1$ then p is the i^{th} smallest element.
- Else if $i \leq |S_1|$ then the i^{th} smallest element is somewhere in S_1. Repeat the process recursively on S_1 looking for the ith smallest element.
- Else i is somewhere in S_2. Repeat the process recursively looking for the $i-|S_1|-1$ smallest element.

Question: How do we select p? Best if p is close to the median. If p is the largest element or the smallest, the problem size is only reduced by 1.

- Always pick the same element, from index n or 1
- Pick a random element
- Pick 3 random elements, and pick the median
- Other method we will see later

Once we have p it is fairly easy to partition the elements:

If A contains: [5 12 8 6 2 1 4 3]

Can create two subarrays, S_1 and S_2. For each element x in A, if x < p put it in S_1 if $x \geq p$ put it in S_2.

p=5
S_1: [2 1 4 3] S_2: [5 12 8 6]

This certainly works, but requires additional space to hold the subarrays. We can also do the partitioning in-place, using no additional space if we maintain pointers starting from the beginning and end of the array as illustrated below:

```
Partition(A,p,r)            ; Partitions array A[p..r]
  x←A[p]                    ; Choose first element as partition element
  i←p-1
  j←r+1
  while true
        do repeat
                j←j-1
           until
                A[j]≤ x
           repeat
                i←i+1
           until A[i]≥x
        if i<j
           then exchange A[i]↔A[j]
           else return j                ; indicates index of partitions
```

Example:

A[p..r] = [5 12 8 6 2 1 4 3]
x=5

	5	12	2	6	2	1	4	3	
i								j	
	5	12	2	6	2	1	4	3	
i								j	
	5	12	2	6	2	1	4	3	
	i							j	
	3	12	2	6	2	1	4	5	swap
	i							j	
	3	12	2	6	2	1	4	5	
	i						j		
	3	12	2	6	2	1	4	5	
		i					j		
	3	4	2	6	2	1	12	5	swap
		i					j		
	3	4	2	6	2	1	12	5	
		i				j			
	3	4	2	6	2	1	12	5	
			i			j			
	3	4	2	6	2	1	12	5	
				i		j			
	3	4	2	1	2	6	12	5	swap
				i		j			
	3	4	2	1	2	6	12	5	
				i	j				
	3	4	2	1	2	6	12	5	
					ij				
	3	4	2	1	2	6	12	5	crossover, i>j
					j	i			

Return j. All elements in A[p..j] smaller or equal to x, all elements in A[j+1..r] bigger or equal to x. (Note this is a little different than the initial example, where we split the sets up into < p, p, and > p. In this case the sets are <=p or >=p. (Consider the case if all array elements are identical). If the pivot point selected happens to be the largest or smallest value, it will also be guaranteed to split off at least one value). This routine makes only one pass through the array A, so it takes time $\Theta(n)$. No extra space required except to hold index variables.

To use this version of Partition in the Selection algorithm, we need to modify the selection algorithm a bit since we are not splitting into < p, p, and > p. Here is the modified algorithm:

; Select from array A, with lower index of p and upper index of r, the i^{th} largest number
Select(A,p,r,i)

```
    If p = r return A[p]
    q ← Partition(A,p,r)          // Q gets the index of where we made the partition
    K ← q – p + 1                 //  Size of left partition
    If i<=K return(Select(A,p,q,i)
    else return(Select(A,q+1,r,i-K)
```

Note the similarity to Quicksoft:

```
QuickSort(A,p,r)
    If p >= r return
    q← Partition(A, p, r)
    QuickSort(A, p, q)
    QuickSort(A, q+1, r)
```

Worst case running time of selection: Pick min or max as partition element, producing region of size n-1.

$$T(n) = T(n-1) + \Theta(n)$$

subprob time to split

Evaluate recurrence by iterative substitution method:

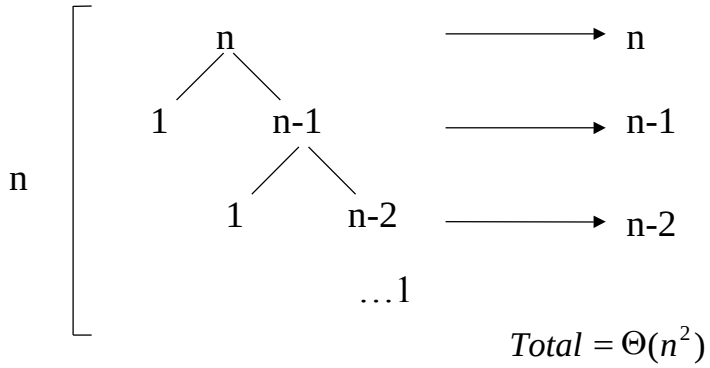
$$T(1) = \Theta(1), T(2) = \Theta(1) + \Theta(2), T(3) = \Theta(1) + \Theta(2) + \Theta(3), \ldots$$

$$\sum_{i=1}^{n} \Theta(i)$$

$$\Theta \sum_{i=1}^{n} i$$

$$= \Theta(n^2)$$

Recursion tree for worst case:



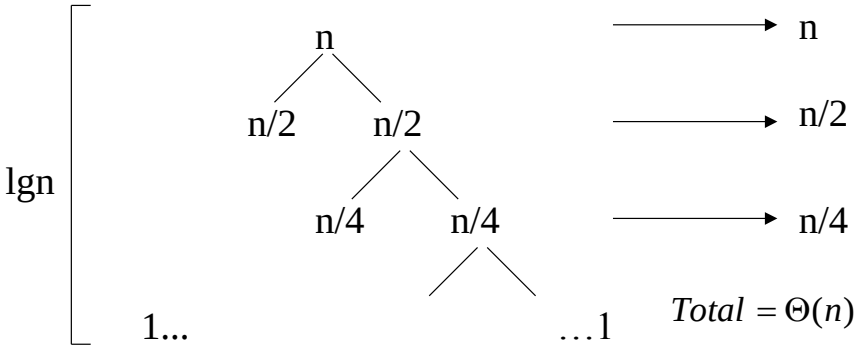
Best-case Partitioning:

In the best case, we pick the median each time.

$$T(n) = T(\frac{n}{2}) + \Theta(n)$$

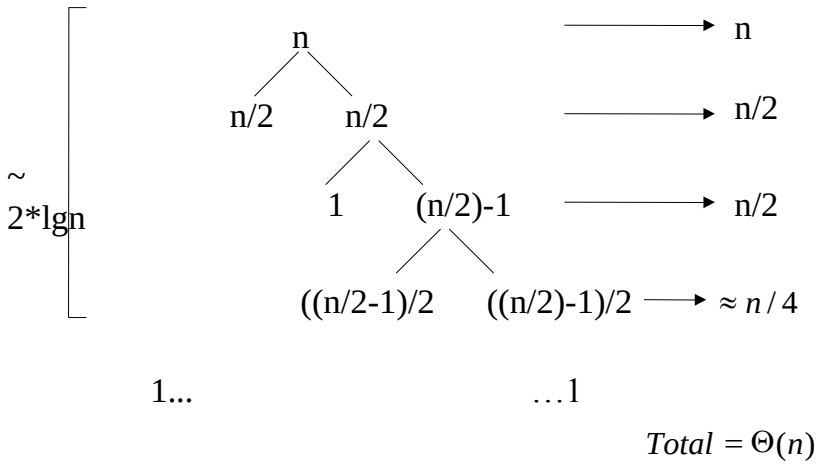
Using the master method: a=1, b=2, f(n)= $\Theta(n)$
Case 3: $\Theta(n)$ is $\Omega(n^{\lg 1})$ and af(n/2) $\leq$ cf(n) since K(n/2) $\leq$ cKn for c = ½ so the solution is f(n)= $\Theta(n)$

Recursion Tree for Best Case:



Average Case: Can think of the average case as alternating between good splits where n is split in half, and bad splits, where a min or max is selected as the split point.

Recursion tree for bad/good split, good split:



$Total = \Theta(n)$

Both are $\Theta(n)$, with just a larger constant in the event of the bad/good split.
So average case still runs in time $\Theta(n)$.

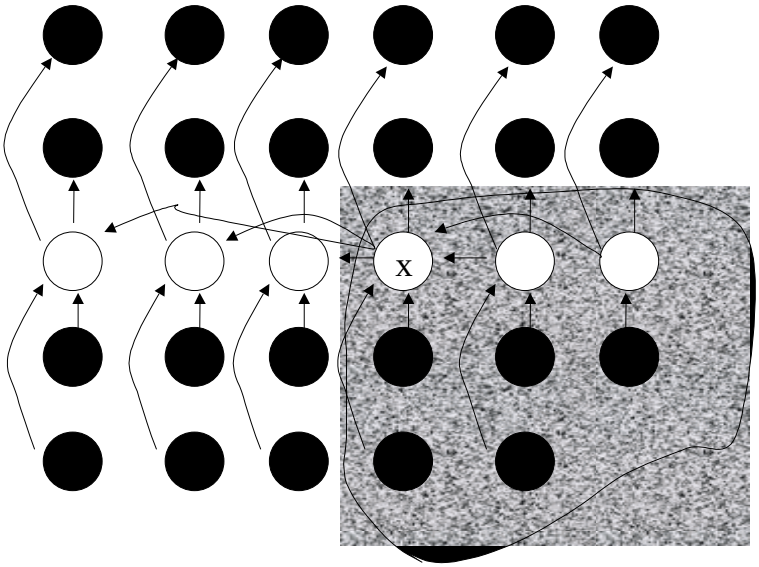
We can solve this problem in worst-case linear time, but it is trickier. In practice, the overhead of this method makes it not useful in practice, compared to the previous method. However, it has interesting theoretical implications.

Basic idea: Find a partition element guaranteed to make a good split. We must find this partition element quickly to ensure $\Theta(n)$ time. The idea is to find the median of a sample of medians, and use that as the partition element.

New partition selection algorithm:

- Arrange the n elements into n/5 groups of 5 elements each, ignoring the at most four extra elements. (Constant time to compute bucket, linear time to put into bucket)
- Find the median of each group. This gives a list M of n/5 medians. (time $\Theta(n)$ if we use the same median selection algorithm as this one or hard-code it)
- Find the median of M. Return this as the partition element. (Call partition selection recursively using M as the input set)

See picture of median of medians:



Guarantees that at least 30% of n will be larger than pivot point p, and can be eliminated each time!

Runtime: $T(n) = T(\frac{n}{5}) + T(\frac{7n}{10}) + O(n)$

select pivot	recurse subprob	overhead of split/select

The O(n) time will dominate the computation resulting in O(n) run time.